

Propagating beliefs in spin glass models

Yoshiyuki Kabashima

Department of Computational Intelligence and Systems Science,

Tokyo Institute of Technology,

Yokohama 2268502, Japan

kaba@dis.titech.ac.jp

Abstract

We investigate dynamics of an inference algorithm termed the belief propagation (BP) when employed in spin glass (SG) models and show that its macroscopic behaviors can be traced by recursive updates of certain auxiliary field distributions whose stationary state reproduces the replica symmetric solution offered by the equilibrium analysis. We further provide a compact expression for the instability condition of the BP's fixed point which turns out to be identical to that of instability for breaking the replica symmetry in equilibrium when the number of couplings per spin is infinite. This correspondence is extended to a SG model of finite connectivity to determine the phase diagram, which is numerically supported.

PACS numbers: 89.90.+n, 02.50.-r, 05.50.+q, 75.10.Hk

Recently, there is growing interest in a similarity between researches on spin glass (SG) and information processing (IP) [1]. Since employment of methods from SG theory provided significant progresses for several problems related to IP such as machine learning [2], error-correcting [3, 4, 5, 6] and spreading codes [7, 8], it is natural to expect that the opposite direction might be possible.

The purpose of this article is to show such an example. More specifically, we show that investigating dynamics of an iterative inference algorithm termed the belief propagation (BP) which has been developed in IP research [9, 10] when employed in SG models provides a new understanding about thermodynamical properties of SG. We show that the replica symmetric (RS) solution known in the equilibrium analysis can be characterized as a *macroscopically* stationary state in BP. We also provide a compact expression of the *microscopic* instability condition around the fixed point in the BP dynamics which turns out to be identical to that of instability for breaking the replica symmetry in equilibrium termed the Almeida-Thouless (AT) instability [11] when the number of connectivity per spin is infinite. Efficacy of this expression for a sparsely connected SG model is also numerically supported.

We here take up a family of Ising SG models define by Hamiltonian

$$\mathcal{H}(\mathbf{S}|\mathbf{J}) = - \sum_{\mu=1}^M J_{\mu} \prod_{l \in \mathcal{L}(\mu)} S_l, \quad (1)$$

where $\mathcal{L}(\mu)$ denotes a set of indices which are connected to a quenched coupling J_{μ} . We assume that each coupling is independently generated from an identical distribution

$$P(J_{\mu}) = \frac{1 + J_0/(\sqrt{C}J)}{2} \delta\left(J_{\mu} - \frac{J}{\sqrt{C}}\right) + \frac{1 - J_0/(\sqrt{C}J)}{2} \delta\left(J_{\mu} + \frac{J}{\sqrt{C}}\right). \quad (2)$$

We further assume that for each μ , $\mathcal{L}(\mu)$ is composed of randomly selected $K \sim O(1)$ spin indices and each spin index l is concerned with C couplings the set of which is denoted as $\mathcal{M}(l)$. $J_0 > 0$ and $J > 0$ are parameters to control the mean and the standard deviation of J_{μ} , respectively, which naturally links the current system (1) to the Sherrington-Kirkpatrick (SK) model [12] in the case of $K = 2$ and $C \sim O(N)$ and to sparsely connected SG models [13, 14, 15] in general.

A major goal of statistical mechanics in the current system is to calculate the microscopic spin average $m_l = \text{Tr}_{\mathbf{S}} S_l \exp[-\beta \mathcal{H}(\mathbf{S}|\mathbf{J})] / \text{Tr}_{\mathbf{S}} \exp[-\beta \mathcal{H}(\mathbf{S}|\mathbf{J})]$ from given Hamiltonian (1). This is formally identical to an inference problem for a posterior distribution $P(\mathbf{S}|\mathbf{J}) \propto \prod_{\mu=1}^M P(J_{\mu}|\mathbf{S})$ derived from a conditional probability $P(J_{\mu}|\mathbf{S}) =$

$\exp [\beta J_\mu \prod_{l \in \mathcal{L}(\mu)} S_l] / \sum_{J_\mu = \pm J / \sqrt{C}} \exp [\beta J_\mu \prod_{l \in \mathcal{L}(\mu)} S_l]$ and a uniform prior, which can be expressed in a bipartite graph as Figure 1 (a). In this expression, spins and couplings are denoted as two different types of nodes and are linked by edges when they are directly connected, which is useful to explicitly represent statistical dependences between estimation variables (spins) and observed data (couplings).

BP is an iterative algorithm defined over the bipartite graph to calculate the spin average for a given set of couplings $\mathbf{J} = (J_\mu)$ [9, 10]. In the current system, this is performed by passing *beliefs* (or messages) between the two types of nodes via edges at each update as

$$\hat{m}_{\mu l}^{t+1} = \tanh \beta J_\mu \prod_{k \in \mathcal{L}(\mu) \setminus l} m_{\mu k}^t, \quad (3)$$

$$m_{\mu l}^t = \tanh \left(\sum_{\nu \in \mathcal{M}(l) \setminus \mu} \tanh^{-1} \hat{m}_{\nu l}^t \right), \quad (4)$$

where beliefs $m_{\mu l}^t$ and $\hat{m}_{\mu l}^t$ are parameters to represent auxiliary distributions at t th update as $P(S_l | \{J_{\nu \neq \mu}\}) = (1 + m_{\mu l}^t S_l) / 2$ and $P(J_\mu | S_l, \{J_{\nu \neq \mu}\}) = \text{Tr}_{S_{k \neq l}} P(J_\mu | \mathbf{S}) P(\mathbf{S} | \{J_{\nu \neq \mu}\}) \propto (1 + \hat{m}_{\mu l}^t S_l) / 2$, respectively. $\mathcal{L}(\mu) \setminus l$ stands for a set of spin indices which belong to $\mathcal{L}(\mu)$ other than l and similarly to $\mathcal{M}(l) \setminus \mu$. Calculating $\hat{m}_{\mu l}$ iteratively, the estimate of the spin average at t th update is provided as

$$m_l^t = \tanh \left(\sum_{\mu \in \mathcal{M}(l)} \tanh^{-1} \hat{m}_{\mu l}^t \right). \quad (5)$$

It is known that BP provides the exact spin average by the convergent solution when the bipartite graph is free from cycles (Figure 1 (b)). Actually, BP is a very similar scheme to the transfer matrix method (TMM) or the Bethe approximation [16, 17] which is frequently used in physics and the current statement can be regarded as a generalization of a known property of TMM that offers the exact results for a one dimensional lattice or a tree. However, BP still has a possibility to introduce something new into physics since it is explicitly expressed as an *algorithm* and such view point has been rare in the research on matters. This strongly motivates us to examine its dynamical properties, which we will focus on hereafter.

Let us first discuss the macroscopic behavior of the BP dynamics (3) and (4). Although the current randomly constructed system is not free from cycles, it can be shown that the typical length of the cycles grows as $O(\ln N)$ with respect to the system size N as long as C is $O(1)$ [18], which implies that the self-interaction from the past state is presumably negligible

in the thermodynamic limit. On the other hand, the self-interaction is also expected as sufficiently small even if C is large since the strength of the coupling becomes weak as $O(C^{-1/2})$. This and eqs. (3) and (4) imply that the time evolution of the macroscopic distributions of beliefs $\pi^t(x) \equiv (1/NC) \sum_{l=1}^N \sum_{\mu \in \mathcal{M}(l)} \delta(x - m_{\mu l}^t)$ and $\hat{\pi}^t(\hat{x}) \equiv (1/NC) \sum_{l=1}^N \sum_{\mu \in \mathcal{M}(l)} \delta(\hat{x} - \hat{m}_{\mu l}^t)$ is likely to be well captured by recursive equations

$$\hat{\pi}^{t+1}(\hat{x}) = \int \prod_{l=1}^{K-1} dx_l \pi^t(x_l) \left\langle \delta \left(\hat{x} - \tanh \beta \mathcal{J} \prod_{l=1}^{K-1} x_l \right) \right\rangle_{\mathcal{J}}, \quad (6)$$

$$\pi^t(x) = \int \prod_{\mu=1}^{C-1} d\hat{x}_{\mu} \hat{\pi}^t(\hat{x}_{\mu}) \delta \left(x - \tanh \left(\sum_{\mu=1}^{C-1} \tanh^{-1} \hat{x}_{\mu} \right) \right), \quad (7)$$

where $\langle \cdots \rangle_{\mathcal{J}}$ represents the average with respect to \mathcal{J} following distribution (2).

The validity of the current argument and its link to the replica symmetric (RS) ansatz in the equilibrium analysis have been shown already for finite C [19, 20]. Here, we further show that these can be extended to the case of infinite C even if the AT stability of the RS solution is broken in equilibrium.

When C becomes infinite, it is more convenient to deal with an auxiliary field of finite strength $h_{\mu l}^t \equiv \sum_{\nu \in \mathcal{M}(l) \setminus \mu} \tanh^{-1} \hat{m}_{\nu l}^t \simeq \sum_{\nu \in \mathcal{M}(l) \setminus \mu} \hat{m}_{\nu l}^t$ rather than $\hat{m}_{\mu l}^t$ since $\hat{m}_{\mu l}^t$ becomes infinitesimal. Due to the central limit theorem, the distribution of the auxiliary field $\rho^t(h) \equiv (1/NC) \sum_{l=1}^N \sum_{\mu \in \mathcal{M}(l)} \delta(h - h_{\mu l}^t)$ can be regarded as a Gaussian

$$\rho^t(h) = \int \prod_{\mu=1}^{C-1} d\hat{x}_{\mu} \hat{\pi}^t(\hat{x}_{\mu}) \delta(h - \sum_{\mu=1}^{C-1} \tanh^{-1} \hat{x}_{\mu}) \simeq \frac{1}{\sqrt{2\pi F^t}} \exp \left[-\frac{(h - E^t)^2}{2F^t} \right], \quad (8)$$

where E^t and F^t are the average and the variance to parameterize the Gaussian distribution $\rho^t(h)$, respectively. The expression in the middle implies $\pi^t(x) = \int dh \rho^t(h) \delta(x - \tanh(h))$. Plugging this into eq. (6) and recursively employing eq. (8), we obtain a compact expression for the update of E^t and F^t as

$$E^{t+1} = \beta J_0 (M^t)^{K-1}, \quad F^{t+1} = \beta^2 J^2 (Q^t)^{K-1}, \quad (9)$$

$$M^t = \int Dz \tanh(\sqrt{F^t} z + E^t), \quad Q^t = \int Dz \tanh^2(\sqrt{F^t} z + E^t), \quad (10)$$

where $Dz \equiv \exp[-z^2/2]/\sqrt{2\pi}$ and M^t and Q^t can be expressed as $M^t \simeq (1/N) \sum_{i=1}^N m_{\mu l}^t \simeq (1/N) \sum_{l=1}^N m_l^t$ and $Q^t \simeq (1/N) \sum_{l=1}^N (m_{\mu l}^t)^2 \simeq (1/N) \sum_{l=1}^N (m_l^t)^2$, respectively, due to the law of large numbers. Eqs. (9) and (10) serve as alternatives of eqs. (6) and (7).

It should be noticed here that these equations can be regarded as the forward iteration of the saddle point equations to obtain the RS solution in the replica analysis of the multi-spin interaction infinite connectivity SG models [1] and, in particular, of the SK model for

$K = 2$ [12]. In order to confirm the validity of the above argument, we compared the time evolution of the belief update (3) and (4) (**BP**) with that of eqs. (9) and (10) (**RS**) for the SK ($K = 2$) model, which is shown in Figures 2 (a) and (b). We also compared them with the trajectory of the naive iteration of the BP's fixed point condition

$$m_l = \tanh \left(\sum_{\mu \in \mathcal{M}(l)} \beta J_\mu \prod_{k \in \mathcal{L}(\mu) \setminus l} m_k - \sum_{\mu \in \mathcal{M}(l)} (\beta J_\mu)^2 \sum_{j \in \mathcal{L}(\mu) \setminus l} \left(\prod_{k \in \mathcal{L}(\mu) \setminus l, j} m_k \right)^2 (1 - m_j^2) m_l \right), \quad (11)$$

(**TAP**) which can be obtained inserting $m_{\mu l} \simeq m_l - (1 - m_l^2) \hat{m}_{\mu l}$ to the fixed point of eqs. (3) and (4) $m_{\mu l}^t = m_{\mu l}$, $\hat{m}_{\mu l}^t = \hat{m}_{\mu l}$ and $m_l^t = m_l$. This becomes identical to the famous Thouless-Anderson-Palmer (TAP) equation of the SK model, in particular, for $K = 2$ [21].

The experiments were performed for $J_0 = 1.5, 0.5$ keeping $J = 1$ and $T = 0.5$, where the AT stability of the RS solution in equilibrium is satisfied for $J_0 = 1.5$ but broken for $J_0 = 0.5$ [11]. Figures 2 (a) and (b) show that **BP** and **RS** exhibit excellent consistency with respect to the macroscopic variables irrespectively of whether the AT stability is satisfied or not. This strongly validates the reduction from BP (3) and (4) to the macroscopic dynamics (9) and (10). On the other hand, **TAP** is considerably different from the others. In a sense, this may be natural because naively iterating eq. (11) is just one of procedures for obtaining a solution and its trajectory in dynamics does not necessarily have any consistency with **BP** or **RS** while the BP's fixed point is correctly characterized by the TAP equation (11) which does have a certain relation to the RS solution in equilibrium as shown in [22]. These figures also imply that the dynamics of BP cannot be traced by a closed set of equations with respect to singly indexed variables m_l^t even for $C \rightarrow \infty$ while the fixed point condition in this limit is provided as coupled equations of m_l (11), which is also observed in a similar system [8].

Although Figures 2 (a) and (b) show that the macroscopic variables rapidly converge to those of the RS solution in BP, this does not imply that BP microscopically converges to a certain solution. In order to probe this microscopic convergence, we examined the squared difference of spin averages between successive updates $D^t \equiv (1/N) \sum_{i=1}^N (m_i^t - m_i^{t-1})^2$, the time evolution of which is shown in the insets of Figures 2 (a) and (b). These illustrate that the (microscopic) local stability of the BP's fixed point can be broken even if the macroscopic behavior seems to converge, which cannot be detected by only examining the reduced macroscopic dynamics (9) and (10).

In order to characterize such instability, we next turn to the stability analysis of the BP updates (3) and (4). Linearizing the updates with respect to the auxiliary field $h_{\mu l} = \tanh^{-1} m_{\mu l}$ around a fixed point solution $m_{\mu l}^t = m_{\mu l}$, we obtain a dynamics of the auxiliary field fluctuation $\delta h_{\mu l}^t$ as

$$\delta h_{\mu l}^{t+1} = \sum_{\nu \in \mathcal{M}(l) \setminus \mu} \frac{\tanh \beta J_{\nu} \prod_{k \in \mathcal{L}(\nu) \setminus l} m_{\nu k}}{1 - \left(\tanh \beta J_{\nu} \prod_{k \in \mathcal{L}(\nu) \setminus l} m_{\nu k} \right)^2} \times \sum_{j \in \mathcal{L}(\nu) \setminus l} \frac{1 - m_{\nu j}^2}{m_{\nu j}} \times \delta h_{\nu j}^t. \quad (12)$$

Analytically solving this linearized equation for a large graph is generally difficult. However, since the current system is randomly constructed, the self-interaction of $\delta h_{\mu l}^t$ from the past can be considered as small as those of beliefs are. This implies that the time evolution of the fluctuation distribution $f^t(y) \equiv (1/NC) \sum_{l=1}^N \sum_{\mu \in \mathcal{M}(l)} \delta(y - \delta h_{\mu l}^t)$ can be provided by a functional equation

$$f^{t+1}(y) = \prod_{\mu=1}^{C-1} \prod_{l=1}^{K-1} dy_{\mu l} f^t(y_{\mu l}) \times \left\langle \delta \left(y - \sum_{\mu=1}^{C-1} \frac{\tanh \beta \mathcal{J}_{\mu} \prod_{k=1}^{K-1} x_{\mu k}}{1 - \left(\tanh \beta \mathcal{J}_{\mu} \prod_{k=1}^{K-1} x_{\mu k} \right)^2} \times \sum_{l=1}^{K-1} \frac{1 - x_{\mu l}^2}{x_{\mu l}} \times y_{\mu l} \right) \right\rangle_{\mathcal{J}_{\mu}, x_{\mu l}}, \quad (13)$$

where $\langle \cdots \rangle_{\mathcal{J}_{\mu}, x_{\mu l}}$ denotes the average over \mathcal{J}_{μ} and $x_{\mu l}$ following eq. (2) and the stationary distribution of $\pi^t(x) = \pi(x)$, respectively, and the stability of the BP's fixed point can be characterized by whether the stationary solution $f^t(y) = f(y) = \delta(y)$ is stable or not in update (13). This formulation makes analytical investigation possible to a certain extent.

In order to connect eq. (13) to the existing analysis, let us first investigate the limit $C \rightarrow \infty$ for which much more results are known compared to the case of finite C . Due to the central limit theorem, the distribution of the field fluctuation can be assumed as a Gaussian $f^t(y) = (1/\sqrt{2\pi b^t}) \exp[-(y - a^t)^2/(2b^t)]$, where a^t and b^t are the mean and the variance of the distribution, respectively. Plugging this expression into eq. (13) offers update rules with respect to a^t and b^t as $a^{t+1} = \beta J_0 M^{K-2} (1 - Q) a^t$ and $b^{t+1} = (\beta J)^2 Q^{K-2} \int Dz \left(1 - \tanh^2(\sqrt{F}z + E) \right)^2 (b^t + (a^t)^2)$, where M, Q, E and F represent the convergent solutions of eqs. (9) and (10). In order to examine the stability of $f(y) = \delta(y)$, we linearize these equations around $a^t = b^t = 0$, which provides the critical condition of the instability with respect to the growth of b^t

$$(\beta J)^2 Q^{K-2} \int Dz \left(1 - \tanh^2(\sqrt{F}z + E) \right)^2 = 1, \quad (14)$$

which becomes identical to that of the AT stability for the infinite range multi-spin interaction SG models and, in particular, for the SK model when $K = 2$ [11]. Furthermore, in the case of the SK model ($K = 2$), the critical condition with respect to a^t around the paramagnetic solution $M = Q = 0$ corresponds to the para-ferromagnetic transition. These mean that the two different phase transitions from the paramagnetic solution can be linked in a unified framework to the dynamical instabilities of BP by eq. (13).

When C is finite, one can numerically perform the stability analysis employing eq. (13), the detail of which will be reported elsewhere. In addition, analytical investigation becomes possible for $K = 2$ as follows since transitions from the paramagnetic solution in this case occur due to the local instability.

For a small β , the paramagnetic solution $\pi(x) = \hat{\pi}(x) = \delta(x)$ ($m_{\mu l} = \hat{m}_{\mu l} = 0$) expresses the correct stable fixed point of the BP dynamics. Inserting this into eq. (13) does not provide a closed set of equations with respect to a finite number of parameters since $f^t(y)$ is no more a Gaussian. However, assuming $f^t(y) \simeq \delta(y)$, the stability analysis can be reduced to coupled equations with respect to the mean and the variance of $f^t(y)$ as $a^{t+1} = (C - 1) \langle \tanh \beta \mathcal{J} \rangle_{\mathcal{J}} a^t$ and $b^{t+1} = (C - 1) \left(\langle \tanh^2 \beta \mathcal{J} \rangle_{\mathcal{J}} b^t + \left(\langle \tanh^2 \beta \mathcal{J} \rangle_{\mathcal{J}} - \langle \tanh \beta \mathcal{J} \rangle_{\mathcal{J}}^2 \right) (a^t)^2 \right)$. Linearizing these around $a^t = b^t = 0$ provides the critical conditions with respect to the growth of a^t and b^t as

$$(C - 1) \langle \tanh \beta \mathcal{J} \rangle_{\mathcal{J}} = 1, \quad (15)$$

$$(C - 1) \langle \tanh^2 \beta \mathcal{J} \rangle_{\mathcal{J}} = 1, \quad (16)$$

respectively. It should be mentioned that a similar condition to eq. (16) was once obtained for a SG model on the Bethe lattice [23] while eq. (15) was not. However, the current scheme may be superior to that employed in [23] as the expression (13) is compact and, therefore, can be easily extended to the case of multi-spin interaction ($K \geq 3$) with the aid of numerical methods while such extension requires higher order perturbation and becomes highly complicated in the other scheme.

Eqs. (15) and (16) might correspond to the para-ferromagnetic and the para-SG phase transitions, respectively, since they do in the limit $C \rightarrow \infty$. In order to examine this, we performed numerical experiments for $N = 2000$ and $C = 4$. Although further investigation may be necessary to prove correctness, the data obtained from 100 experiments of 20000 Monte Carlo steps per spin exhibit good consistency with the analytical expressions (15)

and (16) indicating that the correspondence between the phase transitions in equilibrium and the dynamical instabilities of BP holds for finite C as well (Figure 3).

In summary, we have investigated dynamical behavior of BP when employed in SG models. We have shown that the time evolution of macroscopic variables can be well captured by recursive updates of auxiliary field distributions which becomes identical to the forward iteration of the saddle point equations under the RS ansatz in the replica analysis. We have further shown that the dynamical instability of the BP's fixed point is closely related to the AT instability of the RS solution, which has been numerically supported.

Relationship between the current scheme and an existing AT analysis for finite connectivity SG models [24] that generally requires complicated calculation and is not frequently employed in practice is under investigation. Besides this, extension of the current framework to the replica symmetry breaking (RSB) schemes [25, 26] is a challenging and interesting future work.

This work was partially supported by Grants-in-Aid from the MEXT, Japan, Nos. 13680400, 13780208 and 14084206.

-
- [1] H. Nishimori, Statistical Physics of Spin Glasses and Information Processing – An Introduction, Oxford Press (Oxford), (2001).
 - [2] TLH. Watkin, A. Rau and M. Biehl, Rev. Mod. Phys. **65**, 499 (1993).
 - [3] N. Surlas, Nature **339**, 693 (1989).
 - [4] Y. Kabashima, T. Murayama and D. Saad, Phys. Rev. Lett. **84**, 1355 (2000).
 - [5] N. Nishimori and KYM. Wong, Phys. Rev. E **60**, 132 (1999).
 - [6] A. Montanari and N. Surlas, Eur. Phys. J. B **18**, 107 (2000).
 - [7] T. Tanaka, Europhys. Lett. **54**, 540 (2001).
 - [8] Y. Kabashima, cond-mat/0210535 (2002).
 - [9] J. Pearl, Probabilistic Reasoning in Intelligent Systems: Network of Plausible Inference, Morgan Kaufmann (San Francisco), (1988).
 - [10] DJC. MacKay, IEEE Trans. IT, **45**, 399 (1999).
 - [11] JRL. de Almeida and DJ. Thouless, J. Phys. A **11**, 983 (1978).
 - [12] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).
 - [13] KYM. Wong and D. Sherrington, J. Phys. A **20** L793 (1987).
 - [14] Y. Kabashima and D. Saad, Europhys. Lett. **45**, 97 (1999).
 - [15] T. Murayama and M. Okada, cond-mat/0207637 (2002).
 - [16] HA. Bethe, Proc. R. Soc. London, Ser A **151**, 552 (1935).
 - [17] Y. Kabashima and D. Saad, Europhys. Lett. **44**, 668 (1998).
 - [18] R. Vicente, D. Saad and Y. Kabashima, J. Phys. A **33**, 6527 (2000)
 - [19] T. Richardson and R. Urbanke, IEEE Trans. IT, **47**, 599 (2001).
 - [20] R. Vicente, D. Saad and Y. Kabashima, Europhys. Lett. **51**, 698 (2000).
 - [21] DJ. Thouless, PW. Anderson and RG. Palmer, Phil. Mag. **35**, 593 (1977).
 - [22] M. Mezard, G. Parisi and MA. Virasoro, Spin Glass Theory and Beyond, World Scientific (Singapore), (1986).
 - [23] DJ. Thouless, Phys. Rev. Lett. **56**, 1082, (1986)
 - [24] P. Mottishaw and C. De Dominicis, J. Phys. A **20**, L375, (1987).
 - [25] G. Parisi, J. Phys. A **13**, 1101 (1980).
 - [26] M. Mezard and G. Parisi, Eur. Phys. J. B **20**, 217 (2001).

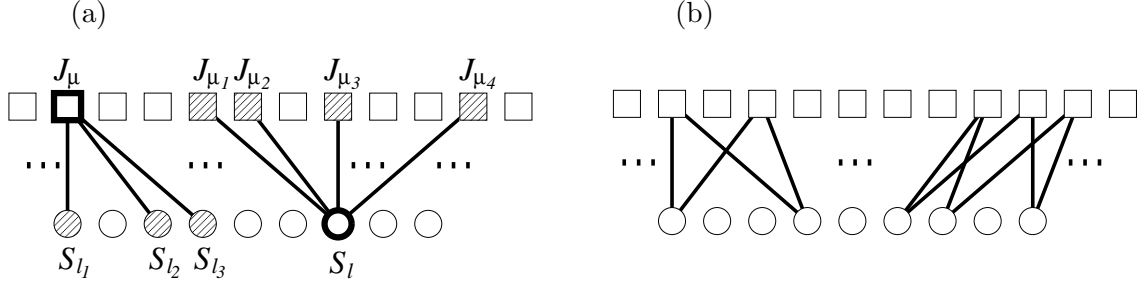


FIG. 1: (a): Graphical expression of SG models in the case of $K = 3$ and $C = 4$. In this expression, each spin S_l denoted as \bigcirc is linked to $C = 4$ couplings J_μ (\square), each of which is connected to $K = 3$ spins. $\mathcal{L}(\mu)$ and $\mathcal{M}(l)$ represent sets of indices of spins and couplings that are related to J_μ and S_l , respectively. In the figure, $\mathcal{L}(\mu) = \{l_1, l_2, l_3\}$ and $\mathcal{M}(l) = \{\mu_1, \mu_2, \mu_3, \mu_4\}$. (b): Cycles in a graph. A cycle is composed of multiple paths to link an identical pair of nodes. It is shown that BP can provide the exact spin averages in a practical time scale if a given graph is free from cycles [9].

(a)

(b)

FIG. 2: Time evolution of macroscopic variables $M^t = (1/N) \sum_{l=1}^N m_l^t$ and $Q^t = (1/N) \sum_{l=1}^N (m_l^t)^2$ in the SK model for the BP updates (3) and (4) (**BP**: \circ), the reduced dynamics (9) and (10) (**RS**: lines) and the naive iteration of the TAP equation (11) (**TAP**: $+$) for (a) $J_0 = 1.5$ and (b) $J_0 = 0.5$ keeping $J = 1$ and $T = 0.5$. **TAP** is plotted only for Q^t in the case of $J_0 = 0.5$ in order to save space. Each marker is obtained from 100 experiments for $N = 1000$ systems. The AT stability is satisfied for $J_0 = 1.5$ but broken for $J_0 = 0.5$. Irrespectively of the AT stability, the behavior of the macroscopic variables in the BP dynamics can be well captured by the reduced dynamics while the naive iteration of the TAP equation does not exhibit any convergence even in the macroscopic scale. Insets: Squared deviation of spin averages between the successive updates $D^t = (1/N) \sum_{l=1}^N (m_l^t - m_l^{t-1})^2$ is plotted for the BP dynamics. The deviation vanishes to zero indicating convergence to a fixed point solution for $J_0 = 1.5$ while remains finite signalling instability of the fixed point for $J_0 = 0.5$.

FIG. 3: Phase diagram for $K = 2$ and $C = 4$ suggested by eqs. (15) and (16). P, F and SG stand for the paramagnetic, the ferromagnetic and the spin glass phases, respectively. The boundary between F and SG is just a conjecture. In order to examine the validity of this diagram, 100 Monte Carlo experiments were performed for $N = 2000$ systems at conditions denoted by +. For each condition, frequencies of macroscopic magnetizations $M_a = (1/N) \sum_{l=1}^N m_l^a$ and overlaps $Q_{ab} = (1/N) \sum_{l=1}^N m_l^a m_l^b$ ($a > b$) were evaluated, where m_l^a is the average of S_l obtained from 20000 Monte Carlo steps per spin for experiments $a, b = 1, 2, \dots, 100$ (insets). For both of the two conditions in P, all of M_a and Q_{ab} fall into the first bin. On the other hand, sharp peaks indicate the order to the ferromagnetic state in F and a broad distribution of Q_{ab} signals the breaking of the replica symmetry in SG.